Seat No. ______________

HB-003-1164003

M. Sc. (Sem. IV) Examination

April - 2023

Mathematics : CMT-4003

(Number Theory-II)

Faculty Code : 003
Subject Code : 1164003

Time : $2\frac{1}{2}$ Hours / Total Marks : **70**

Instructions : (1) There are five questions.
(2) All questions are compulsory.
(3) Each question carries 14 marks.

1 Do as directed : (answer any **seven**) **14**

(a) Find the value of $\langle 0, 3, 3, 3, 3, ..., ...\rangle$ and $\langle -2, 1, 2, 6, 14\rangle$.

(b) Express the numbers $\sqrt{7}+2$ and $\sqrt{3}-1$ in continued fraction expansion.

(c) Write the statement of Hurwitz's Inequality for $x, y > 0$ for Farey fractions.

(d) Find out the values of r_2, r_3 of $\langle 0, 2, 2, 2, 2, 2\rangle$.

(e) Find the period of $\sqrt{18}$.

(f) Find four positive integers for which $1 + 2 + 3 + 4 \dots \dots + n$ is a perfect square.

(g) Find the general solution if any of the equation $131x + 211y = 2$.

(h) If (x, y, z) is a Pythagorean Triplet then show that g.c.d. (y, z) = g.c.d. (x, z).

(i) There is a polynomial $f_n(x)$ with degree n, leading co-efficient 1 and integer coefficients such that $f_n(\sin x) =$ ________, $\forall n \geq 1$ and real x.

(j) Show that, the g.c.d. (x, y) where $\frac{x}{y}$ Farey fraction of the n^{th} row is 1.

2 Answer any two of the following : **14**

(1) Prove that, π is an irrational number.

(2) Prove that, $x^2 - 143y^2 + 1 = 0$ has no solution in integers.

(3) Suppose (x_1, y_1) is the smallest positive solution of $x^2 - dy^2 = 1$ then prove that:

(i) $\left(x_1 + \sqrt{d}\, y_1\right)^n$ is a solution of it for $n \geq 1$.

(ii) Every positive solution is of the form $\left(x_1 + \sqrt{d}\, y_1\right)^n$ for some n.

3 Answer the following: **14**

(1) (i) Suppose r and s are positive integer such that $r > s \geq 1$, $(r, s) = 1$ and r is even then s is odd and vice-versa then prove that, the triplet (x, y, z) is a Primitive Pythagorean triplet where $x = r^2 - s^2$, $y = 2\,rs$ and $z = r^2 + s^2$.

(ii) If $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ be a polynomial with integer coefficient with degree n and if $\frac{a}{b}$ satisfies $f(x)$ then show that, $b \mid a_n$ and $a \mid a_0$ provided $(a, b) = 1$ and $b \geq 0$.

(2) Solve the linear Diophantine equation $172x + 20y = 1000$ by usual method.

OR

3 Answer the following: **14**

(1) Prove that, the value of $f(x) = x^4 + x^3 + x^2 + x + 1$ is a perfect square for $x = -1, 0$ and 3 and for some values of x, $f(x)$ is not a perfect square.

(2) Suppose θ is a quadratic irrational such that $\theta > 1$ and $-1 < \theta' < 0$. Prove that, its periodic continued fraction expansion is purely periodic. Justify the result for $\frac{\sqrt{3}+1}{2}$.

4 Answer the following: **14**

(1) If p, q is a positive solution of $x^2 - ny^2 = 1$, then show that, $\frac{p}{q}$ is a convergent of the continued fraction of $\sqrt{n}$.

(2) (i) Prove that, $x^2 - 80y^2 = -1$ has no solution in integers.

(ii) Prove that, $\lim_{n\to\infty} r_n$ provided its subsequence $\{r_{2n}\}$ has its limit θ, where θ is an irrational. number.

5 Answer any two of the following: **14**

(1) If $a_0, a_1, a_2, ..., a_n, ...$ are integers with $a_i \geq 1; \forall i \geq 1$ then prove that, the number θ is and irrational number if and only if the expansion $\langle a_0, a_1, a_2, ..., a_n, ...\rangle$ is infinite.

(2) If θ is an irrational number and $\frac{a}{b}$ is a rational number then prove that, $b \geq k_{n+1}$ provided $|\theta b - a| < |\theta k_n| - h_n$ for some $n \geq 0$.

(3) Find first four positive solution of $x^2 - 19y^2 = 1$.

(4) Prove that, if x is a rational number then its finite simple continued fraction expansion is always unique provided the last term is not equal to 1.